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Short surface waves in a canal: dependence of frequency on curvature

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Davis has shown by means of a lengthy calculation that, for two-dimensional oscillations in a canal of width $2a$, the mth eigenvalue has the form

$$N_m = \tfrac{1}{2}m\pi - \frac{\lambda_1+\lambda_2}{4m\pi} + o\left(\frac{1}{m}\right),$$

where λ_1/a and λ_2/a are the curvatures of the bounding cross-sectional curve C at its vertical intersections with the free surface. Here the same result is obtained more simply.

1. Introduction and statement of the problem

An infinitely long canal of uniform cross-section is filled with inviscid fluid in such a way that the sides of the canal are vertical at the free surface. Surface tension is neglected and the motion is assumed to be two-dimensional, in planes normal to the generators of the canal; then a velocity potential exists. The motion is assumed to be so small that all the equations can be linearized. Rectangular Cartesian co-ordinates are taken, with the x axis horizontal in a plane of motion and the y axis vertical (y increasing with depth). The origin is taken at a point in the centre of the mean free surface. Then the mean free surface F is given by

$$y = 0, \quad -a < x < a,$$

where $2a$ is the length of F in the x direction. The velocity potential $\phi(x,y,t)$ satisfies

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0 \tag{1.1}$$

in the fluid, and the boundary conditions are

$$\partial\phi/\partial n = 0 \quad \text{on the canal boundary } C$$

(where $\partial/\partial n$ denotes differentiation normal to C) and

$$\partial^2\phi/\partial t^2 - g\,\partial\phi/\partial y = 0 \quad \text{on the mean free surface } F.$$

We shall be concerned with the *normal modes* in which the potential is of the form

$$\phi(x,y,t) = \phi(x,y)\cos\sigma t.$$

The free-surface boundary condition is then

$$K\phi + \partial\phi/\partial y = 0 \quad \text{on} \quad F, \tag{1.2}$$

where $Ka = \sigma^2 a/g\,(= N$, say) is an eigenparameter. It can be shown (see Davis 1965) that there is a non-decreasing enumerably infinite set $N_1, N_2, \ldots$, of positive eigenvalues tending to infinity. (Obviously $N_0 = 0$ is also an eigenvalue.) We shall be concerned with the asymptotic behaviour of N_m when $m \to \infty$, under the additional assumption that the curvatures at the end points $(a, 0)$ and $(-a, 0)$ of C are finite and equal to λ_1/a and λ_2/a respectively.

This problem has been studied by Davis in two papers. In his first paper (Davis 1965) he obtained a suitable integral equation for the values $\phi_m(C)$ of the potential $\phi_m(x, y)$ on the boundary curve C, and inferred that

$$\phi_{2m}(C) = e^{-K_{2m}y}\cos K_{2m}x + \delta_{2m}(x, y) \tag{1.3}$$

(for approximately symmetrical modes) and that

$$\phi_{2m+1}(C) = e^{-K_{2m+1}y}\sin K_{2m+1}x + \delta_{2m+1}(x, y) \tag{1.4}$$

(for approximately antisymmetrical modes), where the harmonic terms $\delta_{2m}(x, y)$ and $\delta_{2m+1}(x, y)$ tend to zero uniformly on C when m tends to infinity. Thus the first terms in (1.3) and (1.4) are dominant near the free surface, but not necessarily elsewhere on C where the first terms are exponentially small. Davis showed that

$$N_m = K_m a = \tfrac{1}{2}m\pi + \epsilon_m, \tag{1.5}$$

where $\epsilon_m \to 0$ when $m \to \infty$, and obtained the stronger result

$$\epsilon_m = O(1/m).$$

The leading terms in (1.3)–(1.5) evidently correspond to the case of parallel walls (Lamb 1932, §228, equations (12), (13)).

In a second paper (Davis 1969) the integral equation was studied in much more detail, and the improved result

$$N_m = \tfrac{1}{2}m\pi - \frac{\lambda_1 + \lambda_2}{4m\pi} + o\left(\frac{1}{m}\right) \tag{1.6}$$

was obtained after a nine-page calculation. A simpler derivation of this result is desirable and will be given in the present note. It will be shown that (1.6) can be deduced from (1.3) and (1.4) with comparatively little effort.

2. Asymptotic calculation of the eigenvalues

Let the potential of the mth normal mode be denoted by $\phi_m(x, y)\cos\sigma_m t$, where $N_m = K_m a = \sigma_m^2 a/g$ is the corresponding eigenvalue. It will be convenient to consider odd and even modes separately; the calculations in the two cases are similar. Let us first consider the even modes, and let us write

$$\Phi_{2m}(x, y) = e^{-K_{2m}y}\cos K_{2m}x; \tag{2.1}$$

cf. (1.3) above. Evidently Φ_{2m} satisfies (1.1) and (1.2). Green's theorem (Lamb 1932, §44, equation (2)) applied to the two harmonic functions ϕ_{2m} and Φ_{2m} states that

$$\int\left(\phi_{2m}(x, y)\frac{\partial\Phi_{2m}}{\partial n}(x, y) - \Phi_{2m}(x, y)\frac{\partial\phi_{2m}}{\partial n}(x, y)\right)ds = 0, \tag{2.2}$$

where the integration is taken along the closed boundary $C+F$, and where $\partial/\partial n$ denotes differentiation normal to the line element ds. Since both ϕ_{2m} and Φ_{2m} satisfy (1.2) we see that there is no contribution from F, and since $\partial\phi_{2m}/\partial n = 0$ on C we see that (2.2) reduces to the equation

$$\int_C \phi_{2m}(x,y)\frac{\partial\Phi_{2m}}{\partial n}(x,y)\,ds = 0, \tag{2.3}$$

where the integration is taken along the curved boundary only. This is the equation that will be used to study N_m.

Since $\Phi_{2m}(x,y)$ contains the exponential factor $e^{-K_{2m}y}$, it follows that $\partial\Phi_{2m}/\partial n$ also contains this factor (see equation (2.5) below); therefore only the neighbourhoods of the two highest points $(a,0)$ and $(-a,0)$ of C contribute effectively to the integral (2.3). We consider these in turn.

Near $(x,y) = (a,0)$ let the equation of C be given parametrically by

$$x-a = \xi(s), \quad y = \eta(s),$$

where s is the arc length along C. Since the curvature is finite, we have

$$\xi = -\frac{1}{2}\frac{\eta^2}{a}\lambda_1 + O\left(\frac{\eta^3}{a^2}\right). \tag{2.4}$$

Then
$$\begin{aligned}\frac{\partial\Phi_{2m}}{\partial n}ds &= \frac{\partial\Phi_{2m}}{\partial\eta}d\xi - \frac{\partial\Phi_{2m}}{\partial\xi}d\eta\\ &= -K_{2m}e^{-K_{2m}\eta}[\cos K_{2m}(a+\xi)\,d\xi - \sin K_{2m}(a+\xi)\,d\eta].\end{aligned} \tag{2.5}$$

From (1.5) we have

$$K_{2m}(a+\xi) = N_{2m} + K_{2m}\xi = m\pi + \epsilon_{2m} + K_{2m}\xi,$$

and
$$\cos K_{2m}(a+\xi) = (-1)^m \cos(\epsilon_{2m}+K_{2m}\xi),$$
$$\sin K_{2m}(a+\xi) = (-1)^m \sin(\epsilon_{2m}+K_{2m}\xi).$$

From (1.3) we therefore have

$$\phi_{2m} = (-1)^m e^{-K_{2m}\eta}\cos(\epsilon_{2m}+K_{2m}\xi) + \delta_{2m}(a+\xi,\eta).$$

The contribution from the neighbourhood of $(a,0)$ to the integral $\int\phi_{2m}(\partial\Phi_{2m}/\partial n)\,ds$ is then

$$-\int K_{2m}e^{-2K_{2m}\eta}\cos(\epsilon_{2m}+K_{2m}\xi)\,[\cos(\epsilon_{2m}+K_{2m}\xi)\,d\xi - \sin(\epsilon_{2m}+K_{2m}\xi)\,d\eta] \tag{2.6}$$

$$+(-1)^{m+1}\int K_{2m}e^{-K_{2m}\eta}\,\delta_{2m}(a+\xi,\eta)\,[\cos(\epsilon_{2m}+K_{2m}\xi)\,d\xi - \sin(\epsilon_{2m}+K_{2m}\xi)\,d\eta]. \tag{2.7}$$

The integral (2.6) is an explicit expression while the integral (2.7) involves the unknown function δ_{2m}. We shall see that the former integral is the dominant one. The integral (2.6) will now be estimated asymptotically for $K_{2m}a$ large and ϵ_{2m} small. (In the calculation the suffix $2m$ will be omitted.) Since $\xi(\eta)$ varies more slowly than η near $\eta = 0$ (see equation (2.4)), the integral can be estimated by expanding all the functions of $K\xi$ in powers of $K\xi$, and then substituting for $\xi(\eta)$ from (2.4). For example,

$$\int e^{-2K\eta}\sin 2K\xi\,d\eta \sim \sum_0^\infty \frac{(-1)^l}{(2l+1)!}\int e^{-2K\eta}(2K\xi)^{2l+1}\,d\eta,$$

of which the leading term is

$$2K\int_0^\infty e^{-2K\eta}\,\xi\,d\eta \sim -\frac{K\lambda_1}{a}\int_0^\infty \eta^2 e^{-2K\eta}\,d\eta$$
$$= -\lambda_1/4K^2a;$$

here the equation

$$\int_0^\infty u^m e^{-u}\,du = m!$$

has been used. It is then not difficult to see that the integral (2.6) has for its leading term the expression

$$-K\int_0^\infty e^{-2K\eta}\,[d\xi-(\epsilon+K\xi)\,d\eta]$$
$$\sim -K\int_0^\infty e^{-2K\eta}\left[-\frac{\eta\lambda_1\,d\eta}{a}-\epsilon\,d\eta+\frac{K\eta^2\lambda_1}{a}\,d\eta\right]$$
$$= \tfrac{1}{2}\epsilon+\lambda_1/8Ka. \tag{2.8}$$

Higher terms can be found if higher terms in the equation (2.4) for $\xi(\eta)$ are known. The integral (2.7) involving δ cannot be explicitly estimated but a bound can be found. Clearly the absolute value of (2.7) is less than

$$K\max_C|\delta(a+\xi,\eta)|\int_0^\infty e^{-K\eta}[|d\xi|+(|\epsilon|+K|\xi|)\,d\eta]$$
$$\leqslant \text{constant}\times K\max|\delta|\int_0^\infty e^{-K\eta}\left(\eta\frac{\lambda_1}{a}+|\epsilon|+\frac{K\eta^2\lambda_1}{2a}\right)d\eta$$
$$= \text{constant}\times\max|\delta|\,(2\lambda_1/Ka+|\epsilon|) = (2\lambda_1/Ka+|\epsilon|)\,o(1), \tag{2.9}$$

since $\delta_{2m}(a+\xi,\eta)\to 0$ when $m\to\infty$. On adding (2.8) and (2.9) it is seen that the contribution to $\int\phi_{2m}(\partial\Phi_{2m}/\partial n)\,ds$ from the neighbourhood of $(a,0)$ is

$$\frac{\epsilon}{2}(1+o(1))+\frac{\lambda_1}{8Ka}(1+o(1)) \tag{2.10}$$

and similarly the contribution from $(-a,0)$ is

$$\frac{\epsilon}{2}(1+o(1))+\frac{\lambda_2}{8Ka}(1+o(1)). \tag{2.11}$$

It follows from (2.3) that the sum of (2.10) and (2.11) must vanish, i.e.

$$\epsilon(1+o(1))+\frac{\lambda_1+\lambda_2}{8Ka}(1+o(1)) = 0,$$

i.e. $$\epsilon_{2m} = -\frac{\lambda_1+\lambda_2}{8K_{2m}a}(1+o(1)) = -\frac{\lambda_1+\lambda_2}{8m\pi}(1+o(1)),$$

which is the relation (1.6) for even m. Similarly the relation can be proved for odd m from the equation

$$\int_C \phi_{2m+1} \frac{\partial \Phi_{2m+1}}{\partial n} ds = 0,$$

where $\Phi_{2m+1}(x, y) = e^{-K_{2m+1} y} \sin K_{2m+1} x$.

3. Discussion

The foregoing calculation has shown how a second asymptotic approximation (1.6) for an eigenvalue can be obtained from a first approximation (1.3) for an eigenfunction. To obtain the same result, Davis (1969) went to a second approximation for the eigenfunction, and thence to a second approximation for the eigenvalue. He found an ambiguity in the sign of the correction and determined this by a separate argument. In the present note there is no ambiguity in sign.

The use made of Green's theorem in the present work may be compared with its earlier use in wave-making and scattering problems, where a good approximation for wave-making and transmission coefficients was obtained from a comparatively crude approximation for the potential. (See in particular, Ursell 1961, §4.) In these problems the second harmonic function in Green's theorem was chosen to approximate as closely as possible to the *Green's function* for the problem, and thus had a physical interpretation. In our present problem there is no obvious reason for choosing Φ_m as the second function.

It was assumed that the curvature at both ends of the boundary C is finite and not zero, $\xi/a \sim \text{constant} \times (\eta/a)^2$. A similar calculation can be made when $\xi/a \sim \text{constant} \times (\eta/a)^l$ near both ends, for any $l > 1$.

REFERENCES

DAVIS, A. M. J. 1965 Two-dimensional oscillations in a canal of arbitrary cross-section. *Proc. Camb. Phil. Soc.* **61**, 827–846.

DAVIS, A. M. J. 1969 Short surface waves in a canal: dependence of frequency on curvature. *Proc. Roy. Soc.* A **313**, 249–260.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

URSELL, F. 1961 The transmission of surface waves under surface obstacles. *Proc. Camb. Phil. Soc.* **57**, 638–668.